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A Trichotomy: Interactions of Factors Prolonging Sequential and Concurrent Mental Processes in Stochastic Discrete Mental (PERT) Networks

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Suppose the mental processes required for performing a task are partially ordered, so that some pairs of processes are sequential and some are concurrent. Then they can be represented in a directed acyclic network, a PERT network. Suppose the duration of each process is a nonnegative random variable. Suppose two experimental factors are available, each selectively prolonging a different process by adding a nonnegative random variable to its duration. Sternberg (1969) pointed out that if all the processes are in series, the factors will have additive effects on reaction time. Here we show that if the factors affect concurrent processes, the factors will have subadditive effects. Subadditive effects are also possible if the prolonged processes are sequential, but in a Wheatstone bridge. If the PERT network has no subnetwork homeomorphic to a Wheatstone bridge, then ineractions between factors prolonging sequential processes will be nonnegative, and in practice will often be positive. The results are illustrated in a detailed analysis of a particular network, the Embellished Wheatstone Bridge.

A human performing an information processing task is carrying out a multitude of mental processes which are organized somehow. There are many theories about the details. One dichotomy of theoretical interest is whether the processes are sequential or concurrent; another is whether a process must be completed before its successors can start; another is whether the output of a process is a discrete entity, such as a number or symbol, or a continuous quantity, such as a strength. Theories are formed by taking a choice on some of these issues, allowing both options on other issues, and being silent about others.

This paper develops further the theory of mental processes in a special arrangement, a directed acyclic network (e.g., Fig. 1). This arrangement allows for both sequential and concurrent processes, assumes that a process must be completely finished before its successors can start, and makes no assumption about the nature of the output of a process. The preceeding paper (Townsend & Schweickert, 1989) has more details.

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As cognitive models, directed acyclic networks are of an intermediate level of complexity. They are more general than the models considered by Donders (1868) and Sternberg (1969), in which all the processes are in series. They differ from parallel distributed processing models (Hinton & Anderson, 1981; McClelland & Rumelhart, 1986; Rumelhart & McClelland, 1986) in which assumptions are made about the "microstructure" of processing. Because PERT networks allow a moderately complicated system to be represented without excessive attention to detail, they are the major class of models considered in the mathematical theory of scheduling (Coffman, 1976). They are one of the standard ways to model systems of jobs, such as those encountered by the operating system of a computer (Coffman & Denning, 1973) or a manager in a factory.

Evidence that mental processes are arranged in PERT networks in certain tasks is discussed in Schweickert (1978, 1980, 1983a) for experiments on the Stroop effect, sentence verification, and the psychological refractory period, carried out by various investigators and by Schweickert. The major objection to PERT networks as models for human information processing is the requirement, said to be implausible, that a process cannot start before all its immediate predecessors have finished. Miller (1988) points out that there is, as yet, no compelling evidence rejecting this assumption as a basis for functioning. The assumption is very plausible for tasks requiring symbol manipulation. For example, in mental arithmetic, it is quite likely that the process of adding the units column is completed before the process of adding the 10's column is started.

The additive factor method. Sternberg's (1969) additive factor method assumes, as does Donders' (1868) method, that all the processes are in series. Suppose there are two experimental factors each of which prolongs a different process in the series. For example, suppose making the stimulus dimmer prolongs a perceptual process and increasing the number of alternatives prolongs a decision process. Then the increase in reaction time produced by prolonging both processes will be the sum of the effects of prolonging the processes individually.

Sternberg (1969) also said that if the two factors interact, it is likely that they both affect the same process in the series. Taylor (1976) expanded on this theme. Here, we argue that if the two factors interact, it may be that the assumption that all the processes are in series is violated.[1] We further suggest that it is not merely the occurrence of an interaction that is important, but whether the interaction is positive or negative. By a positive interaction, or superadditivity, we mean that the combined effect of the factors is larger than the sum of their individual effects; a negative interaction, or subadditivity, is defined analogously. The preceding paper compares our assumptions with earlier approaches in more completeness.

[1] Another possibility is that the processes are indeed in series, but a process begins execution before its successor is finished. This possibility will not be dealt with here. McClelland (1979), Townsend and Ashby (1983), pp. 401–412), Ashby (1982), and Schweickert (in press) have made progress on theoretically analyzing such systems and Miller (1982a, 1982b) and Meyer, Yantis, Osman, and Smith (1985) have suggested tests of certain versions of these.

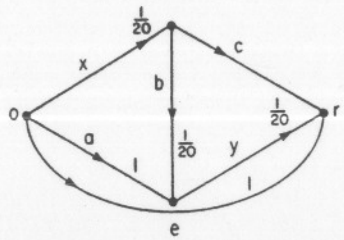

FIG. 1. An Embellished Wheatstone Bridge. Each arc represents a mental process. The mean baseline duration of each process is indicated by a number on the corresponding arc.

An example. When not all the processes are in series, interactions between factors can come and go, depending on seemingly unrelated aspects of the experimental situation. Suppose two laboratories investigate the effects of two experimental factors in the same task. The experiments are done in the same way, except that in the first laboratory trials began without a warning signal, while in the second a warning signal was used. The hypothetical data from laboratory 1, in the upper panel of Table 1, have a negative interaction and the data from laboratory 2, in the lower panel, have a positive interaction.

Many post hoc explanations could be offered for these results. However, a unified explanation suffices. They were generated with the arrangement of processes in Fig. 1. The processes have independent exponential distributions, with means as indicated in the figure. Factor 1 changes the mean duration of process x from 1/20 to 1, factor 2 does the same for process y. The warning signal is assumed to shorten process c, which might be concerned with response preparation. The PERT network in Fig. 1 is called an Embellished Wheatstone Bridge (EWB). The values in Table 1, and in the other examples in this paper, were calculated by Donald L.

TABLE 1

Hypothetical Response Time Data from Two Laboratories

Level of factor 1	Level of factor 2: 1	2
	Laboratory One[a]	
1	1.868	2.457
2	2.457	3.008
	Laboratory Two[b]	
1	1.528	2.254
2	1.890	2.689

[a] Negative interaction: process c has mean 1

[b] Positive interaction: process c has mean 1/20.

Fisher (personal communication, August 1986) using the algorithm of Fisher and Goldstein (1983). Further development of algorithms is in Fisher, Saisi, and Goldstein (1985) and Vorberg (1988).

We now proceed to our results on interactions. Ultimately we will show that the qualitative trichotomy of superadditivity, additivity, and subadditivity is sufficient to assess much that is of interest about the mental network underlying a task.

Terminology

For the reader's convenience, a brief summary of the notation and concepts of stochastic discrete mental networks is given here. The terms are fully defined in the preceding paper (Townsend & Schweickert, 1989); see also Schweickert (1978, 1982).

Suppose each process x which must be executed to perform a task is represented by an arc from starting vertex x' to finishing vertex x'' in a directed, acyclic network with source o and sink r. A process begins execution when and only when all its adjacent preceding processes are finished. The duration of process x is a nonnegative random variable $D(x)$, with finite mean. A value taken on by $D(x)$ is denoted $d(x)$.

Suppose the duration $D(x)$ of each process x has taken on a value $d(x)$. In other words, suppose each arc has associated with it some nonnegative real number. For a given assignment of nonnegative real numbers to the arcs, the *duration of a path* is the sum of the durations of all the processes on it. We let $d(a, b)$ denote the duration of the longest path from vertex a to vertex b. Suppose process y follows process x on a path. The *slack* from process x to process y is $s(xy') = d(o, y') - d(o, x') - d(x) - d(x'', y')$. The *total slack* for process x is $s(xr) = d(o, r) - d(o, x') - d(x) - d(x'', r)$. Given values of $s(xr)$, $s(yr)$ and $s(xy')$, we let $s(yx'') = s(yr) - s(xr) + s(xy')$.

When the durations of the processes are random variables, then so are the path durations and the slacks. Let the random duration of the longest path from vertex a to vertex b be denoted $D(a, b)$. The random variables for the slacks $S(xr)$, $S(xy')$ $S(yr)$, and $S(yx'')$ are defined by substituting in the above equations the appropriate values $D(a, b)$ in place of $d(a, b)$ for every pair of vertices $\langle a, b\rangle$.

Prolonging Processes. Suppose x and y are two processes in a directed, acyclic network. If there is a path from the finishing vertex of x to the starting vertex of y, or vice versa, then x and y are *sequential.* Otherwise they are *concurrent.* Suppose factor 1 affects process x and factor 2 affects process y. Let $D(z)$ denote the duration of process z when both factors are at their lowest levels. In particular, the duration of x is $D(x)$ and that of y is $D(y)$ when both factors are at their lowest levels. Suppose when factor 1 is at level 2, the duration of process x is $D(x) + U$, where U is a nonnegative random variable. Factor 1 is said to *increment the duration* of x. The preceding paper (Townsend & Schweickert, 1989) shows that the

assumption that a factor increments the duration of the process it prolongs is equivalent to the assumption that the factor orders the distributions of the process it prolongs. Suppose the random variable corresponding to every process duration except that of x is unchanged when the level of factor 1 is changed. Suppose when factor 2 is at level 2, the duration of process y is $D(y) + V$, where $D(y)$ is as defined before and V is a nonnegative random variable. The effect of factor 2 is to increment the duration of y. Suppose the random variable corresponding to the duration of every process except y is unchanged when the level of factor 2 is changed.

Let the random variable T_{ij} be the time to complete the task when factor 1 is at level i and factor 2 is at level j. A particular value of T_{ij} is denoted t_{ij}. Given a set of real number values $d(or)$, $s(xr)$, $s(yr)$, u, and v (with $s(xy')$ included if x precedes y), the quantities t_{11}, t_{12}, t_{21}, and t_{22} are produced.[2] We are interested in the expected value of the expression $A = T_{22} - T_{21} - T_{12} + T_{11}$; $E[A] = E[T_{22}] - E[T_{21}] - E[T_{12}] + E[T_{11}]$.

When x and y are concurrent, the slacks $S(xr)$ and $S(yr)$ are linear combinations of the durations $D(or)$, $D(ox')$, $D(x)$, $D(x''r)$, $D(oy')$, $D(y)$, and $D(y''r)$. These durations and the nonnegative random variables U and V have a joint density, $g(d(or), ..., d(y''r), u, v)$. It follows that the slacks $S(xr)$, $S(yr)$, and the random variables U, V, and $T_{11} = D(or)$ have an induced joint density $f(s(xr), s(yr), u, v, t_{11})$. When x and y are sequential, with, say, x preceding y, there is an analogous density for the random variables $S(xy')$, $S(xr)$, $S(yr)$, U, V, and T_{11}.

[2] One of the reviewers argues that the random variable $A = T_{22} - T_{12} - T_{21} + T_{11}$ is not well defined, because the T_{ij} are defined on different sample spaces. His position is, "I am not saying that the theorem is incorrect but only that the proof is not watertight." As we understand his objection, it is that in a typical reaction time experiment it is not possible to obtain a set of observations $\{t_{11}, t_{21}, t_{12}, t_{22}\}$ as defined here, because on any given trial the subject is observed in exactly one of the experimental conditions, not all four of them. This objection would have implications about the sample estimator of the population value $E[T_{22} - T_{21} - T_{12} + T_{11}]$. Since our theorems concern the population value, the comment, as we understand it, is not directly relevant to the theorems.

There seem to be two slightly different ways to respond in more detail to this query. (1) Let T'_{11}, T'_{21}, T'_{12}, T'_{22} denote the times to complete the task in the four conditions in the event that all are observed on the same occasion. Let T''_{ij} denote the time to complete the task in condition (i, j) when the task completion time for that condition only is observed on an occasion, as in a typical reaction time experiment. Then the investigator will never know whether $E[T'_{ij}] = E[T''_{ij}]$, since the two ways to make the observations are mutually exclusive. The preceding equation must be assumed in order to apply our results to a typical reaction time experiment, but we think most investigators will find the assumption innocuous, even from this viewpoint. (2) From a slightly different perspective, it is really only necessary to consider A as an expression to be integrated, not necessarily a random variable. The regrouping of terms in the proofs of the theorems is justified by the standard tenets of the integral calculus—once the condition of Definition 10 from the preceding paper is in place. Furthermore, in every case, an alternate, sometime more tedious, proof can be carried out even without the regrouping. Thus, assuming Definition 10 of the first paper, the integrand of $E(T_{ij})$ $(i, j = 1, 2)$ in the first part of the proof actually refer to the same (respectively speaking) variables t_{11}, u, v, $s(xr)$, and so on. It can then be seen that $E(t_{11})$ will cancel out and that $E[\mathrm{Max}\{[u - s(xr)]^{-}, [v - s(yr)]^{+}\}]$ must be less than $E\{[u - s(xr)]^{-}\} + E\{[v - s(yr)]^{+}\}$, thus leading to the negative contrast.

The major assumptions underlying the theorems on stochastic discrete mental processes can be stated informally as follows. Processe are arranged in a PERT network. One experimental factor prolongs a process x in the network, and another prolongs a process y. Each factor prolongs the process associated with it by incrementing the duration of the process, leaving the durations of the other processes unchanged. When the level of a factor is changed, there is no change in the graph underlying the PERT network. A joint density exists for the random variables $D(or)$, $S(xr)$, $S(yr)$, U, and V (with $S(xy')$ included if x precedes y). Certain marginal densities are used in the proofs; these are assumed to exist and to be well defined (marginal selectivity; see preceding paper).

Otherwise, there are no assumptions about the form of the distributions, or about the independence of the random variables. In particular, even if each subject has a different joint distribution for the random variables, the theorems would apply to the expected values of the reaction times, where the expectation is with respect to the mixture distribution produced by all the subjects.

The Qualitative Trichotomy and Its Implications

We begin with networks in which processes x and y are concurrent, and then discuss the more complicated case where they are sequential.

Concurrent Processes

An example of a network in which x and y are concurrent is in Fig. 2. Examples of the behavior of factors prolonging x and y are in Tables 2 and 3. In these examples, it is assumed that the process duration have exponential distributions and are pairwise independent. The mean duration for each process (the reciprocal of the rate) is given in the figure. Evidence that some mental processes have exponential distributions is given in Ashby (1982), Ashby and Townsend (1980), Kohfeld (1969), and Kohfeld, Santee, and Wallace (1981a, 1981b). Ratcliff (1988) argues that nonexponential processes may sometimes statistically approximate certain predictions by exponential processes.

In the first example, in Table 2, the factor prolonging x is assumed to do so by decreasing the rate parameter of x from 20 to 1. Likewise, the factor prolonging y

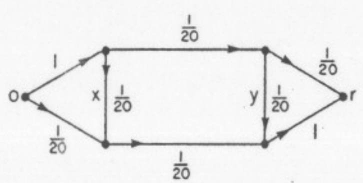

FIG. 2. When x and y are concurrent, factors selectively prolonging them will interact. The number on each arc is the mean of the exponential distribution assumed for the baseline in the calculations reported in the text.

TABLE 2

Effects of Factors Selectively Prolonging Concurrent Processes x and y in Fig. 2 by Decreasing Their Rate Parameters from 20 to 1. The Number in Each Cell is the Mean Reponse Time in Arbitrary Units.

Level of factor 1	Level of factor 2: 1	2
1	2.139	3.054
2	3.054	3.551

is assumed to decrease its rate parameter for y from 20 to 1. The resulting reaction times lead to a negative interaction of $-.418$.

In the second example, in Table 3, the factor prolonging x is assumed to insert an additional independent exponentially distributed process in series with the original process x, so the new process x has a generalized gamma distribution (McGill & Gibbon, 1965). The inserted process has a rate parameter of 1. Likewise, the factor prolonging y is assumed to insert a new independent exponentially distributed process in series with y. The rate parameter of the new process is 1. The resulting reaction times lead to a negative interaction of $-.463$.

Townsend and Ashby (1983; see also Townsend, 1984) proved that a large number of parallel models cannot predict additivity and that the class of independent parallel models must predict subadditivity. The following theorem generalizes the subadditivity classification to an even larger set of models, where independence is not required. It states that if processes x and y are concurrent and a restricted range of values for the prolongations is not used, then the interaction, as given by $E[A]$, is negative.

The proof is a variation on that in Townsend and Schweickert (1985). It is given in some detail; later proofs will be briefer. Let $R = \{\langle s(xr), s(yr), u, v\rangle\}$ be the set

TABLE 3

Effects of Factors Selectively Prolonging Concurrent Processes x and y as in Fig. 2 by Concatenating a Process with Mean 1 in Series with Each of Them

Level of factor 1	Level of factor 2: 1	2
1	2.139	3.102
2	3.102	3.602

of values taken by the random vector $\langle S(xr), S(yr), U, V\rangle$. Let $R_N \subset R$ be the region where the prolongations are larger than the slacks, that is

$$R_N = \{\langle s(xr), s(yr), u, v\rangle \mid u > s(xr) \text{ and } v > s(yr)\}.$$

For a real number r, let

$$[r^+]\begin{cases} = 0 & \text{if } r \leqslant 0 \\ = r & \text{if } r > 0. \end{cases}$$

THEOREM 1 (Townsend & Schweickert, 1985). *If processes x and y are concurrent, then $E[A] \leqslant 0$. If the probability that $\langle s(xr), s(yr), u, v\rangle$ takes on values in R_N is positive, then $E[A] < 0$.*

Proof. In Schweickert (1978) it is shown that for given values $\langle s(xr), s(yr), u, v\rangle$, if x and y are concurrent,

$$t_{21} = t_{11} + [u - s(xr)]^+$$
$$t_{12} = t_{11} + [v - s(yr)]^+$$
$$t_{22} = t_{11} + \max\{[u - s(xr)]^+, [v - s(yr)]^+\}.$$

Then

$$a = t_{22} - t_{12} - t_{21} + t_{11}$$
$$= \max\{[u - s(xr)]^+, [v - s(yr)]^+\} - [u - s(xr)]^+ - [v - s(y)]^+.$$

Clearly, $a = 0$ when and only when $u \leqslant s(xr)$ or $v \leqslant s(yr)$. Otherwise $a < 0$.

Let the joint density of $s(xr), s(yr), u, v, t_{11}$ be $f(s(xr), s(yr), u, v, t_{11})$. Let the joint density of $s(xr), s(yr), u, v$ be $f_{1234}(s(xr), s(yr), u, v)$; let the joint density of $s(xr), u, t_{11}$ be $f_{135}(s(xr), u, t_{11})$; let the joint density of $s(yr), v, t_{11}$ be $f_{245}(s(yr), v, t_{11})$; and let the density of t_{11} be $f_5(t_{11})$. Then

$$E[A] = E[T_{22}] - [E[T_{21}] - E[T_{12}] + E[T_{11}]$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (t_{11} + \max\{[u - s(xr)]^+, [v - s(yr)^+\})$$
$$\times f(s(xr), s(yr), u, v, t_{11})\, ds(xr)\, ds(yr)\, du\, dv\, dt_{11}$$
$$- \int_0^\infty \int_0^\infty \int_0^\infty (t_{11} + [u - s(xr)]^+) f_{135}(s(xr), u, t_{11})\, ds(xr)\, du\, dt_{11}$$
$$- \int_0^\infty \int_0^\infty \int_0^\infty (t_{11} + [v - s(yr)]^+) f_{245}(s(yr), v, t_{11})\, d(yr)\, dv\, dt_{11}$$
$$+ \int_0^\infty t_{11} f_5(t_{11})\, dt_{11}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (t_{11} + \max\{[u - s(xr)]^+, [v - s(yr)]^+\}$$
$$- t_{11} - [u - s(xr)]^+$$
$$- t_{11} - [v - s(yr)]^+ + t_{11}) f(s(xr), s(yr), u, v, t_{11})$$
$$\times ds(xr)\, ds(yr)\, du\, dv\, dt_{11}$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (\max\{[u - s(xr)]^+, [v - s(yr)]^+\}$$
$$- [u - s(xr)]^+ - [v - s(yr)]^+) f_{1234}(s(xr), s(yr), u, v)$$
$$\times ds(xr)\, ds(yr)\, du\, dv$$
$$= \iiiint_{R - R_N} 0 f_{1234}(s(xy), s(yr), u, v)\, ds(xr)\, ds(yr)\, du\, dv$$
$$+ \iiiint_{R_N} (\max\{u - s(xr), v - s(yr)\} - (u - s(xr)) - (v - s(yr))$$
$$\times f_{1234}(s(xr), s(yr), u, v)\, ds(xr)\, ds(yr)\, du\, dv.$$

The result follows immediately. Q.E.D.

Sequential processes

The story would be simpler if factors prolonging sequential processes always had nonnegative interactions, but they do not. If the network is a Bare Wheatstone Bridge (BWB), as in Fig. 3, the interactions between factors prolonging x and y can be negative. If the baseline durations of x and y are large, the interaction will approach additivity, but it can never be positive.

Two graphs are said to be *homeomorphic* if one can be transformed to the other by repeated application of the following two procedures: (a) an arc is replaced by two adjacent arcs in series and (b) two adjacent arcs in series are replaced by a single arc. Negative interactions for factors prolonging sequential processes can only occur if the task network has a subnetwork homeomorphic to the Bare Wheatstone Bridge (Schweickert, 1978).

The Bare Wheatstone Bridge is important for another reason. A directed acyclic graph (DAG) is *series–parallel* if it can be constructed recursively as follows. (1) A single directed arc between two vertices is a series–parallel DAG. (2) (Serial composition) If a directed arc is inserted as an adjacent predecessor to the starting vertex of a series–parallel DAG, or inserted as an adjacent successor to the terminating vertex of a series–parallel DAG, the resulting DAG is series–parallel. (3) (Parallel composition) If a directed arc is inserted with its starting vertex at the starting vertex of a DAG and with its terminal vertex at the terminal vertex of a DAG, the resulting DAG is series–parallel.

TABLE 4

Effects of Factors Selectively Prolonging x and y in a Bare Wheatstone Bridge as in Fig. 3 by Decreasing Their Rate from Parameters 20 to 1.

Level of factor 1	Level of factor 2: 1	1
1	1.552	2.267
2	2.267	2.889

An important result is that a directed acyclic graph is series-parallel if and only if it does not contain a subgraph homeomorphic to the Bare Wheatstone Bridge (Dodin, 1985; Kaerkes & Mohring, 1978; Valdes, Tarjan, & Lawler 1979).

Tables 4 and 5 show the results of prolonging processes in the Bare Wheatstone Bridge illustrated in Fig. 3. The number on each arc gives the mean duration of each process. The processes are assumed to have pairwise independent exponential distributions.

In Table 4, each of processes x and y is prolonged by having its rate parameter changed from 20 to 1. The interaction is $-.093$. In Table 5, process x is prolonged by having an independent exponentially distributed process inserted in series with x. The new process has rate parameter 1, and is independent of every other process in the network. Process y is prolonged in the same way. (The processes inserted to prolong x and y are independent of each other as well.) The interaction is $-.102$.

The next theorem shows that the interaction will tend to be negative if the network is a Bare Wheatstone Bridge with x and y as in Fig. 3.

Let the joint density of $\langle S(xy'), S(xr), S(yr), U, V\rangle$ be $f(s(xy'), s(xr), s(yr), u, v)$. Let $R = \{\langle s(xy'), s(xr), s(yr), u, v\rangle\}$ be the space of values taken on by the random vector $\langle s(xy'), s(xr), s(yr), u, v\rangle$. The space R can be divided into 18 regions as in Table 6, from Schweickert (1978, 1982).

For the Bare Wheatstone bridge, regardless of the distributions of the process

TABLE 5

Effects of Factors Selectively Prolonging x and y Arranged in a Bare Wheatstone Bridge as in Fig. 3 by Concatenating a Process with Mean 1 in Series with Each of Them.

Level of factor 1	Level of factor 2: 1	2
1	1.552	2.304
2	2.304	2.954

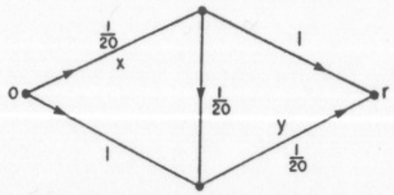

FIG. 3. Processes x and y are on opposite sides of a Wheatstone Bridge. For the calculations reported in the text, each process is assumed to have an exponential distribution whose mean is given by the number associated with each arc.

durations, the only regions possible are R_1 through R_9, since $s(xr) - s(xy) \leq 0$ (Schweickert, 1978, 1982). If the prolongations are big enough to have an effect, $a < 0$.

THEOREM 2. *Suppose the task network is a Bare Wheatstone Bridge with x and y on opposite sides of the bridge, as in Fig. 3. Then $E[A] \leq 0$. If the pertinent random variables $S(xy')$, $S(xr)$, $S(yr)$, U, V take on values in regions R_5, R_6, R_8, R_9 with nonzero probability, then $E(A) < 0$.*

Proof. We have

$$\begin{aligned} E[A] &= \iiiint\!\!\int_R af(s(xy'), s(xr), s(yr), u, v) \\ &\quad \times ds(xy')\, ds(xr)\, ds(yr)\, du\, dv \\ &= \iiiint\!\!\int_{R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_7} 0 f(s(xy'), s(xr), s(yr), u, v) \\ &\quad \times ds(xy')\, ds(xr)\, ds(yr)\, du\, dv \\ &\quad + \iiiint\!\!\int_{R_5 \cup R_6 \cup R_8 \cup R_9} af(s(xy'), s(xr), s(yr), u, v) \\ &\quad \times ds(xy')\, ds(xr)\, ds(yr)\, du\, dv \\ &\leq 0 \end{aligned}$$

since $a < 0$ in regions R_5, R_6, R_8, and R_9. The second part of the theorem follows immediately. Q.E.D.

We now consider the case in which x precedes y, but no subnetwork of the task network is homeomorphic to a Wheatstone Bridge with x and y on opposite sides of the bridge. In that case, the only regions of R possible are $R10$ through $R18$ (Schweickert, 1978, 1982). The following theorem states that in that case, the interactions are nonnegative.

TABLE 6

t_{21}, t_{12}, t_{22} and a as a Function of u and v for Sequential Processes x and y.

	u	v	t_{21}	t_{12}	t_{22}	a
			$k = s(xr) - s(xy') = s(yr) - s(yx'') \leqslant 0$			
R_1	$u \leqslant s(xr) \leqslant s(xy')$	$v \leqslant s(yr) \leqslant s(yx'')$	t_{11}	t_{11}	t_{11}	0
R_2	$s(xr) \leqslant u \leqslant s(xy')$	$v \leqslant s(yr) \leqslant s(yx'')$	$u - s(xr) + t_{11}$	t_{11}	$u - s(xr) + t_{11}$	0
R_3	$s(xr) \leqslant s(xy') \leqslant u$	$v \leqslant s(yr) \leqslant (yx'')$	$u - s(xr) + t_{11}$	t_{11}	$u - s(xr) + t_{11}$	0
R_4	$u \leqslant s(xr) \leqslant s(xy')$	$s(yr) \leqslant v \leqslant s(yx'')$	t_{11}	$v - s(yr) + t_{11}$	$v - s(yr) + t_{11}$	0
R_5	$s(xr) \leqslant u \leqslant s(xy')$	$s(yr) \leqslant v \leqslant s(yx'')$	$u - s(xr) + t_{11}$	$v - s(yr) + t_{11}$	$\max\{u - s(xr), v - s(yr)\} + t_{11}$	$\max\{s(xr) - u, u(yr) - v\}$
R_6	$s(xr) \leqslant s(xy') \leqslant u$	$s(yr) \leqslant v \leqslant s(yx'')$	$u - s(xr) + t_{11}$	$v - s(yr) + t_{11}$	$u - s(xr) + t_{11}$	$s(yr) - v$
R_7	$u \leqslant s(xr) \leqslant s(xy')$	$s(yr) \leqslant s(yx'') \leqslant v$	t_{11}	$v - s(yr) + t_{11}$	$v - s(yr) + t_{11}$	0
R_8	$s(xr) \leqslant u \leqslant s(xy')$	$s(yr) \leqslant s(yx'') \leqslant v$	$u - s(xr) + t_{11}$	$v - s(yr) + t_{11}$	$v - s(yr) + t_{11}$	$s(xr) - u$
R_9	$s(xr) \leqslant s(xy') \leqslant u$	$s(yr) \leqslant s(yx'') \leqslant v$	$u - s(xr) + t_{11}$	$v - s(yr) + t_{11}$	$u + v - s(yr) - s(xy') + t_{11}$	$s(xr) - s(xy')$
			$k = s(xr) - s(xy') = s(yr) - s(yx'') \geqslant 0$			
R_{10}	$u \leqslant s(xy') \leqslant s(xr)$	$v \leqslant s(yx'') \leqslant s(yr)$	t_{11}	t_{11}	t_{11}	0
R_{11}	$s(xy') \leqslant u \leqslant s(xr)$	$v \leqslant s(yx'') \leqslant s(yr)$	t_{11}	t_{11}	t_{11}	0
R_{12}	$s(xy') \leqslant s(xr) \leqslant u$	$v \leqslant s(yx'') \leqslant s(yr)$	$u - s(xr) + t_{11}$	t_{11}	$u - s(xr) + t_{11}$	0
R_{13}	$u \leqslant s(xy') \leqslant s(xr)$	$s(yx'') \leqslant v \leqslant s(yr)$	t_{11}	t_{11}	t_{11}	0
R_{14}	$s(xy' - \leqslant u \leqslant s(xr)$	$s(yx'') \leqslant v \leqslant s(yr)$	t_{11}	t_{11}	$[v - s(yr) + u - s(xy')]^{+} + t_{11}$	$[v - s(yr) + u - s(xy'')]^{+}$
R_{15}	$s(xy') \leqslant s(xr) \leqslant u$	$s(yx'') \leqslant v \leqslant s(yr)$	$u - s(xr) + t_{11}$	t_{11}	$u - s(xr) + v - s(yx'') + t_{11}$	$v - s(yx'')$
R_{16}	$u \leqslant s(xy') \leqslant s(xr)$	$s(yx'') \leqslant s(yr) \leqslant v$	t_{11}	$v - s(yr) + t_{11}$	$v - s(yr) + t_{11}$	0
R_{17}	$s(xy') \leqslant u \leqslant s(xr)$	$s(yx'') \leqslant s(yr) \leqslant v$	t_{11}	$v - s(yr) + t_{11}$	$v - s(yr) + u - s(xy') + t_{11}$	$u - s(xy')$
R_{18}	$s(xy') \leqslant s(xr) \leqslant u$	$s(yx'') \leqslant s(yr) \leqslant v$	$u - s(xr) + t_{11}$	$v - s(yr) + t_{11}$	$u + v - s(yr) - s(xy') + t_{11}$	$s(xr) - s(xy')$

In Townsend and Schweickert (1985) it was shown that in the special case of the network consisting of x and y in series with an arc z concurrent with both x and y, $E[A] > 0$. The following theorem considers the general case.

THEOREM 3. *Suppose process x precedes process y, and there is no subnetwork of the task network homeomorphic to a Wheatstone bridge with x and y on opposite sides of the bridge. Then $E[A] \geqslant 0$. If every region of R possible under the hypotheses has a nonzero probability*[3] *then $E[A] > 0$.*

Proof. We have

$$\begin{aligned} E[A] &= \iiiint\!\!\!\int_R af(s(xy'), s(xr), s(yr), u, v) \\ &\quad \times ds(xy')\, ds(xr)\, ds(yr)\, du\, dv \\ &= \iiiint_{R_{14} \cup R_{15} \cup R_{17} \cup R_{18}} af(s(xy'), s(xr), s(yr), u, v) \\ &\quad \times ds(xy')\, ds(xr)\, ds(yr)\, du\, dv \\ &\geqslant 0 \end{aligned}$$

since $a > 0$ in regions R_{14}, R_{15}, R_{17}, and R_{18}. The second part follows immediately. Q.E.D.

COROLLARY. *If $E[A] < 0$, then either x and y are concurrent or there is a subnetwork of the task network homeomorphic to a Wheatstone bridge with x and y on opposite sides of the bridge.*

Proof. The conclusion follows immediately.

The following two theorems provide information about the conditions under which additivity and superadditivity arise.

THEOREM 4. *If $E[A] > 0$, then there is a path from o to r not containing x or y.*

Proof. Since $E[A] > 0$, x and y are sequential. Without loss of generality, assume x precedes y. Assume, contrary to the conclusion, that every path from o to r contains x. Then for every set of values taken on by the path durations, $s(xr) = 0$. Then $s(xr) - s(xy') \leqslant 0$, so $E[A] \leqslant 0$ contrary to the hypothesis. Hence, there is a path from o to r not containing x. Likewise, there is a path from o to r not containing y.

Suppose every path from o to r not containing x contains y, and every path from o to r not containing y contains x. Then either x or y is on the critical path, so

[3] From Table 6, it is apparent that a slightly stronger result holds. If the random variables $S(xy')$, $S(xr)$, $S(yr)$, U, V take on values in R_{15}, R_{17}, or R_{18} with nonzero probability, or if $[v - s(yr) + u - s(xy')]^{+} > 0$ with nonzero probability (in region R_{14}), then $E[A] > 0$.

$s(xr) = 0$ or $s(yr) = 0$. In either case, $s(xr) - s(xy') \leq 0$ so $E[A] \leq 0$ contrary to the hypothesis. Hence there exists a path from o to r not containing x or y. Q.E.D.

The following terminology is needed for the next theorem. When the levels of the factors prolonging processes x and y are chosen, there is a corresponding density $f(s(xy'), s(xr), s(yr), u, v)$. If for all such choices $E[A] = 0$, we write $E[A] \equiv 0$. The duration of a particular process z may be so small that it is never on a critical path. The *criticality index* of a process z is the probability that z is on a critical path in a sample of process durations. A process never on a critical path has criticality index 0.

THEOREM 5. *Suppose there is no upper bound on the value the duration of a process can have. Suppose all possible regions in R have nonzero probability of occurring. If $E[A] \equiv 0$, then there is no arc z concurrent with both x and y and having a positive criticality index.*

Proof. Since $E[A]$ is 0, by Theorem 1 x and y are sequential. Without loss of generality, assume x precedes y. Since for a given set of process durations

$$s(xy') = d(oy') - d(ox'') - d(x) - d(x''y')$$

and there is no upper bound on the duration of x, there exist values of $d(x)$ large enough to make $s(xy') = 0$. Suppose $s(xy') = 0$.

Suppose there exists an arc z concurrent with x and y, with positive criticality index. The total slack for z is $s(zr) = d(or) - d(oz') - d(z) - d(z''r)$. Since z has nonzero criticality index, there exist values of $d(z)$ large enough to make $s(zr) = 0$. Since z is on none of the three paths in the equation for $s(xy')$, the duration of $s(xy')$ is not affected by the value of $d(z)$. Suppose z is critical while $s(xy') = 0$.

Process x cannot be on a critical path with z, since x does not precede or follow z. If there is only one critical path, x is not on it, so $s(xr) > 0$. Suppose there are more than one critical path, and x is on one of them. This path does not contain z, and since there is no upper bound on the duration of z, a slightly larger value for $d(z)$ will make this path noncritical. Then $s(xr) > 0$.

Since $s(xr) > 0$, while $s(xy') = 0$, region R_{18} is a possible region, and has some nonzero probability of occurring. If $E[A] = 0$, the contributions from the regions R_5, R_6, R_8, and R_9, where the interaction is negative, must exactly equal the contributions from the regions R_{14}, R_{15}, R_{17}, and R_{18}, where the interaction is positive. Clearly this will not happen for all choices of the joint density $f(s(xy), s(sr), s(yr), u, v)$, hence it is not the case that $E[A] \equiv 0$. Then no process z exists as described. Q.E.D.

APPLICATION OF THEOREMS TO THE EMBELLISHED WHEATSTONE BRIDGE

To use these results to analyze a task, the investigator would use experimental factors to prolong processes individually and in pairs. The interactions would be classified as positive, negative, or zero, and various classes of networks would be selected for further consideration, or eliminated. If it is known for every pair of processes whether the pair is sequential or concurrent, then the set of all possible digraphs can be constructed by using the transitive orientation algorithm; see Golumbic (1980) or Schweickert (1983b) for details.

An investigator would not ordinarily have information about all pairs of processes, but might, by some combination of empirical knowledge, common sense, and previous analysis of reaction times, be able to restrict his attention to a set of candidate networks. To illustrate the use of the theorems in this paper, we suppose the investigator has decided by some means that the relevant network is an Embellished Wheatstone Bridge or a subnetwork of it, and consider how the results of prolonging two processes, x and y, further restrict the possibilities.

The Embellished Wheatstone Bridge (EWB), see Fig. 1, provides a prototype of sorts for all three types of qualitative mean reaction time behavior, additivity, superadditivity, and subadditivity. It can do this because it contains the four fundamental classes of paths: (1) paths containing both x and y; (2) paths containing x and not y; (3) paths containing y and not x; (4) paths containing neither x nor y. Depending on which paths are dominant, in terms of tendencies toward long processing times, the EWB can act like (a) the Bare Wheatstone Bridge (BWB) in Fig. 3; (b) a network in which x and y are concurrent; (c) a network in which x and y lie on a path but are concurrent with an additional path, and paths of type (2) and (3) are absent; (d) a network in which x and y are in series.

It is relatively obvious from the preceding sections that (a) and (b) produce subadditivity, (c) produces superadditivity, and (d) produces additivity. Slightly more subtle is the other direction: What may we conclude from super-, sub-, or additivity discovered in our data, when we are willing to confine ourselves to the EWB, or subnetworks of an EWB? It turns out that quite strong conclusions may be drawn within this class of systems.

I. If additivity is found for all levels of u and v, then by Theorems 1, 2, and 5 the bridge in the EWB is present (b in Figure 1), the path containing neither x nor y is absent (e) in Figure 1), and it is not the case that both c and a are present (on the paths containing, respectively, x but not y and y but not x). Fig. 4 shows three mental networks satisfying these conditions.

II. If superadditivity is found, then by Theorems 1 and 4, (a) x and y are connected on some path (i.e., the bridge, b, is present) and (b) the path that contains neither x nor y is present (e). The other paths, a and c, may or may not be present. A BWB is ruled out because it is incapable of producing superadditivity. Figure 5 indicates the appearance of such networks. Observe that the EWB can (but need not) produce such behavior and the simplest network evidencing this type of behavior is simply serial in x, y with a concurrent path e.

III. If subadditivity is found, then by Theorems 1 and 2 and EWB could (but need not) be responsible. The behavior of the network is dominated by either the

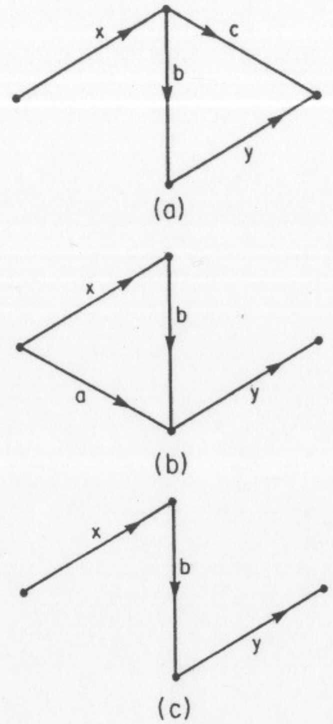

FIG. 4. If the task network is known to be a subnetwork of the Embellished Wheatstone Bridge, and additivity is found for all levels of the factors prolonging x and y, then the network is one of the three illustrated here.

BWB or a subnetwork of he EWB in which x and y are concurrent. Two possibilities for the latter are shown in Fig. 6b and 6c. Continued prolongation of x and y will lead to asymptotic additivity in case of an EWB or BWB but subadditivity persists always if a network in which x and y are concurrent is responsible for the original subadditivity.

Thus, additivity suggests a network in which the various subprocesses can be segregated into two subnetworks of activity, one containing x and the other y, that are connected through a single vertex. The subnetwork can be identified as an "x" subnetwork because its overall processing duration is selectively affected by the experimental factor. The same can be said with regard to y in the second subnetwork, so overall we see a sort of serial macro-system involving separate subsystems for x and y.

The full EWB can, as we might expect, produce either super- or subadditivity but not additivity except by a balancing act. The BWB can only produce subadditivity. Networks in which x and y are concurrent can only produce subadditivity. Super-

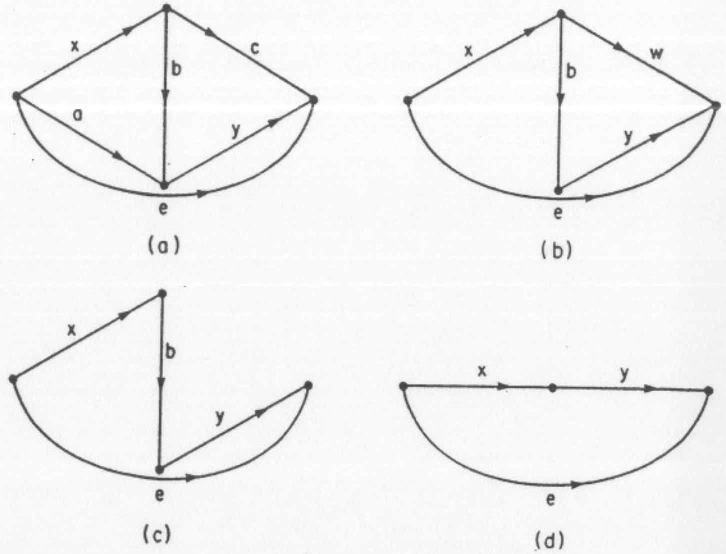

FIG. 5. Networks producing superadditivity.

additivity implies the presence of process e, as in the special case where x and y are serial in concurrence with e. Finally, any network in which there exists a path connecting x and y (i.e., b) can aporoach additivity asymptotically if the baseline durations of x and y, $D(x)$ and $D(y)$, are relatively long.

Incidentally, it is interesting that one of the present class of mental networks cannot, as x and y (alone) are manipulated, act subadditively for a while and then alter to acting superadditively or vice versa. For instance, when certain of the above subadditive networks approach additivity asymptotically, they do not go through an intermediate phase of superadditivity. This is one indication that our method is based on a falsifiable theory, albeit a quite general one. Strong empirical evidence that such alternating super- and subadditive phases occurred throughout the reasonable range of x and y manipulations would falsify the entire class of PERT networks.

Finally, it is possible to gain more identification of a network by manipulating other path durations. For example, in the event of superadditivity associated with EWB, one can learn if a and c are present because if they are prolonged, the network will be pushed into subadditivity. Similar remarks hold in the case of subadditivity. Ultimately, then, there is hope of almost complete identification of the underlying mental network employing factorial reaction time experiments within our methodology and within the general class of EWB's. The only real exception is in certain cases such as x, y in series where their order would not be determined by our method. However, in many such cases, common sense may dictate the order (also see Schweickert, 1983b).

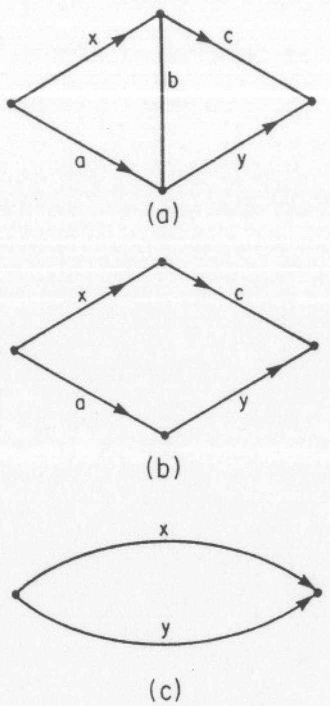

FIG. 6. Networks producing subadditivity.

Epilogue. This paper presents two types of theorems. Constructing networks from data requires theorems of the form: If property P holds for the interactions, then property Q holds for the network. A complete set of such theorems would dictate the set of all networks possible for a given set of data. The set of theorems in this paper is, in this sense, incomplete. Some of the theorems presented are of the form: If property Q holds for the network, then property P holds for the interactions. These shed some light on the set of networks possible for a given set of data, but may be of greater use as steps toward further theorems of the first type. In any case, the applications to EWB's may be considered complete within that realm.

REFERENCES

ASHBY, F. G. (1982a). Deriving exact predictions from the cascade model. *Psychological Review*, **89**, 599–607.

ASHBY, F. G. (1982b). Testing the assumptions of exponential, additive reaction time models. *Memory and Cognition*, **10**, 125–134.

ASHBY, F. G., & TOWNSEND, J. T. (1980). Decomposing the reaction time distribution: Pure insertion and selective influence revisited. *Journal of Mathematical Psychology*, **21**, 93–123.

COFFMAN, E. G. (1976). *Computer and job-shop scheduling theory*. New York: Wiley.

COFFMAN, E. G. & DENNING, P. J. (1973). *Operating systems theory*. Englewood Cliffs, NJ: Prentice-Hall.

DODIN, B. (1985). Reducibility of stochastic networks. *Omega International Journal of Management Science*, **13**, 223–232.

DONDERS, F. C. (1868). Die Schnelligkeit Psychischer Processe. *Archiv für Anatomie und Physiologie*. 657–681. [On the speed of mental processes. In W. G. Koster (Ed. and Trans.), *Attention and performance II*. Amsterdam: North-Holland. Reprinted from *Acta Psychologica*, 1969, **30**].

FISHER, D. L., & GOLDSTEIN, W. M. (1983). Stochastic PERT networks as models of cognition: Derivation of the mean, variance and distribution of reaction time using order-of-processing (OP) diagrams. *Journal of Mathematical Psychology*, **27**, 121–151.

FISHER, D. L., SAISI, D., & GOLDSTEIN, W. M. (1985). Stochastic PERT networks: OP diagrams, critical paths, and the project completion time. *Computers and Operations Research*, **12**, 471–482.

GOLUMBIC, M. C. (1980). *Algorithmic graph theory and perfect graphs*. New York: Academic Press.

HINTON, G. E., & ANDERSON, J. A. (1981). *Parallel models of associative memory*. Hillsdale, NJ: Erlbaum.

KAERKES, R., & MOHRING, R. H. (1978). *Vorlesungen über Ordnungen und Netzplantheorie* (Schriften zur Informatik und Angewandten Mathematik 45). Aachen, Federal Republic of Germany: Technischen Universität Aachen.

KOHFELD, D. L. (1969). Effects of the intensity of auditory and visual ready signals on simple reaction time. *Journal of Experimental Psychology*, **82**, 88–95.

KOHFELD, D. L., SANTEE, J. L., AND WALLACE, N. D. (1981a). Loudness and reaction time, I. *Perception and Psychophysics*, **29**, 535–549.

KOHFELD, D. L., SANTEE, J. L., AND WALLACE, N. D. (1981b). Loudness and reaction time. II. Identification of detection components at different intensities and frequencies. *Perception and Psychophysics*, **29**, 550–562.

MCCLELLAND, J. L. (1979). On the time relations of mental processes: An examination of processes in cascade. *Psychological Review*, **86**, 287–330.

MCCLELLAND, J. L., & RUMELHART, D. E. (1986). *Parallel distributed processing: Explorations in the microstructure of cognition* (Vol. 2). Cambridge, MA: MIT Press.

MCGILL, W. J., & Gibbon, J. (1965). The general-gamma distribution and raction times. *Journal of Mathematical Psychology*, **2**, 1–18.

MEYER, D. E., YANTIS, S., OSMAN, A. M., & SMITH, J. E. K. (1985). Temporal properties of human information processing: Tests of discrete vs continuous models. *Cognitive Psychology*, **17**, 445–518.

MILLER, J. (1982a). Discrete versus continuous stage models of human information processing: In search of partial output. *Journal of Experimental Psychology: Human Perception and Performance*, **8**, 273–296.

MILLER, J. O. (1982b). Effects of noise letters on decisions: discrete or continuous flow of information? *Perception and Psychophusics*, **31**, 227–236.

MILLER, J. (1988). Discrete and continuous models of human information processing: Theoretical distinctions and empirical results. *Acta Psychologica*, **67**, 191–257.

RATCLIFF, R. (1988). A note on mimicking additive reaction time models. *Journal of Mathematical Psychology*, **32**, 192–204.

RUMELHART, D. E., & MCCLELLAND, J. (1986). *Parallel distributed processing: Explorations in the microstructure of cognition* (Vol. 1). Cambridge, MA: MIT Press.

SCHWEICKERT, R. (1978). A critical path generalization of the additive factor method: Analysis of a Stroop task. *Journal of Mathematical Psychology*, **18**, 105–139.

SCHWEICKERT, R. (1980). Critical path scheduling of mental processes in a dual task. *Science*, **209**, 704–706.

SCHWEICKERT, R. (1982). The bias of an estimate of coupled slack in stochastic PERT networks. *Journal of Mathematical Psychology*, **26**, 1–12.

SCHWEICKERT, R. (1983a). Latent network theory: Scheduling of processes in sentence verification and the Stroop effect. *Journal of Experimental Psychology: Learning, Memory and Cognition*, **9**, 353–383.

SCHWEICKERT, R. (1983b). Synthesizing partial orders given comparability information: Partitive sets and slack in critical path networks. *Journal of Mathematical Psychology*, **27**, 261–276.

SCHWEICKERT, R. (1985). Separable effects of factors on speed and accuracy: Memory scanning, lexical decision, and choice tasks. *Psychological Bulletin*, **97**, 530–546.

SCHWEICKERT, R. (in press). Separable effects of factors on activation functions in discrete and continuous models: *d'* and evoked potentials. *Psychological Bulletin.*

STERNBERG, S. (1969). The discovery of processing stages: Extensions of Donders' method. In W. G. Koster (Ed.), *Attention and performance II* (pp. 276–315). Amsterdam: Elsevier/North-Holland.

TAYLOR, D. A. (1976). Stage analysis of reaction time. *Psychological Bulletin*, **83**, 161–191.

TOWNSEND, J. T. (1984). Uncovering mental processes with factorial experiments. *Journal of Mathematical Psychology*, **28**, 363–400.

TOWNSEND, J. T., & ASHBY, F. G. (1983). *The stochastic modeling of elementary psychological processes.* New York: Cambridge Univ. Press.

TOWNSEND, J. T., & SCHWEICKERT, R. (1985). Interactive efects of factors affecting processes in stochastic PERT networks. In G. d'Ydewalle (Ed.), *Cognition, information processing, and motivation.* Amsterdam: North-Holland.

TOWNSEND, J. T., & SCHWEICKERT, R. (1989). Toward the trichotomy method of reaction times: Laying the foundation of stochastic mental networks. *Journal of Mathematical Psychology*, **33**, 309–327.

VALDES, J., TARJAN, R. E., & LAWLER, E. L. (1979). The recognition of series-parallel digraphs. *Proceedings of 11th Annual Association for Computing Machinery Symposium on Theory of Computing* (pp. 1–12). Association for Computing Machinery.

VORBERG, D. (1988). Network models of reaction time. Paper presented at Twenty-fourth International Congress of Psychology. In S. W. Link and H. C. Miko (Conveners), *Response Latency: Theories of Psychological Processes.* Sydney, Australia.

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